BETH'S THEOREM IN CARDINALITY LOGICS

BY

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ABSTRACT

We prove that the Beth definability theorem fails for a comprehensive class of first-order logics with cardinality quantifiers. In particular, we give a counterexample to Beth's theorem for $L(Q)$, which is finitary first-order logic (with identity) augmented with the quantifier "there exists uncountably many".

0. Introduction

The Beth definability theorem is a basic theorem about L-finitary predicate calculus with identity. It asserts the natural closure condition on a logic: that implicit definitions made in the logic can be replaced by explicit ones. For which natural logics extending L that are currently under investigation, does Beth's theorem hold?

Barwise [1] shows that Beth's theorem holds for the first-order logic based on any admissible subset of HC. (Actually the Craig interpolation theorem is proved; any logic obeying Craig's theorem also obeys Beth's.) Gregory [3], using results of Morley, proves that Beth's theorem fails for any logic between $\mathscr{L}_{\omega_2\omega}$ and $\mathscr{L}_{\infty\omega}$. Malitz [6] proves that Beth's theorem fails for any logic between $\mathscr{L}_{\omega_1\omega_1}$ and $\mathscr{L}_{\infty\infty}$.

This paper is devoted to counterexamples for Beth's theorem in first order logics based on cardinality quantifiers. For each ordinal α, let $L(Q_\alpha)$ be finitary first order logic with identity and the additional quantifier $(Q_\alpha x)$ with the interpretation "there are at least ω_α many". Let $L(Q)$ be finitary first order logic with identity

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and the additional quantifier (Qx) with the interpretation "there are as many x as there are elements in the model". Let $L^{-}(Q_\alpha)$, $L^{-}(Q)$ respectively be the same as $L(Q_\alpha)$, $L(Q)$ except that identity is not allowed.

Yasuhara [7] shows that Beth's theorem holds for $L^{-}(Q)$ and for $L^{-}(Q_\alpha)$, provided ω is singular.† Once again, this is proved via Craig's theorem.

We prove that Beth's theorem fails for $L(Q)$, and for every $L(Q_\alpha)$. We also prove that for regular ω_α, Beth's theorem fails for $L^{-}(Q_\alpha)$.

We present out results in a very general form. We define the infinitary first order logic L^* which encompasses any first order logic with cardinal quantifiers ever presented. We prove that Beth's theorem fails for any logic between (1) $L(Q)$ and L^*, (2) $L(Q_\alpha)$ and L^*, for any $\alpha > 0$, and (3) $L^{-}(Q_\alpha)$ and L^*, if ω_α is regular, $\alpha > 0$.

The hypothesis of Beth's theorem is that every structure has at most one expansion satisfying ϕ. Let us call "weak Beth's theorem" the statement obtained by replacing the hypothesis of Beth's theorem with the stronger hypothesis: every structure has exactly one expansion satisfying ϕ. Little is known about weak Beth's theorem. Does it hold in $L_{\infty\omega}$ or $L(Q_1)$?

1. Back and forth through L^*

Below, we will carefully define the logic L^* which encompasses any proposed first order logic with cardinality quantifiers. L^* will be the least logic containing the atomic formulae of L, closed under $\sim$, $\exists$, and each Q_α, and most crucially: the conjunction of any class of formulae of L^* of size at most that of V = class of all sets, is a formula of L^* (provided there are at most finitely many free variables).

We first define some concepts underlying the semantics of all logics discussed in this paper. We use the n-ary relation symbol R^n_m, $0 < n, m$. Since we are presenting counterexamples, nothing is lost by omitting constant and function symbols. We use $=$ for identity.

A signature is a finite set of relation symbols. A σ-structure consists of a nonempty domain D together with an assignment to each n-ary relation symbol in σ, an n-ary relation on D. We require that D be a *set* (as opposed to a class).

To define L^*, first assign (in a one-one fashion) a set which is not a sequence, to each atomic formula of L and to each of the signs $\&$, $\sim$, $\exists x_n$, $Q_\alpha x_n$, $1 \leqq n$, α an ordinal (in V). Let $\lfloor y$ be the set associated with y.

† Actually, Yasuhara deals only with infinite models. The counter examples in this paper will remain counterexamples if finite models are likewise omitted from consideration.

A tree will be a *class* of nonempty finite sequences of sets, closed under initial segments. The formulae of L^* will be certain well-founded trees. If s is a finite sequence and x a set, let xs be the sequence obtained by appending x at the front of s, and let $\langle x\rangle$ be the sequence of length 1 consisting of x. Let $\langle x,y\rangle = x\langle y\rangle$.

We now inductively define the collection of formulae of L^*, as well as their free variables. The free variables of a collection of formulae are just the free variables of its elements.

If ϕ is an atomic formula of L, then $\{\langle |\phi| \rangle\}$ is a formula of L^*. This formula is written ϕ. The free variables of ϕ are exactly the variables occurring in ϕ in the sense of L.

If ϕ is a formula of L^* then so is $\{|\sim| s : s \in \phi\} \cup \{|\sim|\}$. This formula is written $(\sim \phi)$. The free variables of $(\sim \phi)$ are exactly those of ϕ.

If ϕ is a formula, then $\{|\exists x_n| s : s \in \phi\} \cup \{|\exists x_n|\}$, and $\{|Q_\alpha x_n|\, s : s \in \phi\} \cup \{|Q_\alpha x_n|\}$ are formulae of L^*. These formulae are written $(\exists x_n)(\phi)$, $(Q_\alpha x_n)(\phi)$ respectively. The free variables of either formula are exactly the free variables of ϕ minus x_n.

If F is a partial function from V into formulae of L^*, then $\{|\&|(xs) : s \in F(x), x \in \mathrm{Dom}(F)\} \cup \{\langle|\&|, x\rangle : x \in \mathrm{Dom}(F)\} \cup \{\langle |\&| \rangle\}$ is a formula of L^*, provided there are at most finitely many free variables in $\mathrm{Rng}(F)$. This formula is written $\&(F)$. The free variables of $\&(F)$ are exactly the free variables of $\mathrm{Rng}(F)$.

The semantics of L^* is defined in the straightforward way, where $\sim$ is negation, $\&$ is conjunction, $\exists x_n$ means "there is an x_n", and $Q_\alpha x_n$ means "there are at least ω_α many x_n".

In this paper, the logic $L(Q_\alpha)$ has the same syntax as the $L(Q)$ of Keisler [4], except with Q_α in place of Q; $Q_\alpha x_n$ is interpreted as "there are at least ω_α many x_n". The logic $L(Q)$ here will have the same syntax as the $L(Q)$ of Keisler [4]; Qx_n is interpreted as "there are just as many x_n as there are elements in the model". Take $L^-(Q)$, $L^-(Q_\alpha)$ to be $L(Q)$, $L(Q_\alpha)$ respectively, without equality.

It is clear that $L(Q)$ and each $L(Q_\alpha)$ are sublogics of L^*, in the sense that every elementary class of the former is an elementary class of the latter.

We use $\equiv$ for elementary equivalence in L^*. We will use only the three relation symbols: e for R_0^2, r for R_1^2 and p for R_0^1. We will use only the two signatures $\{e,r\}$ and $\{e,r,p\}$. We write structures of the first signature as (D,E,R), and of the second as (D,E,R,P).

Let L' be logic between L without identity, and L^*.† Beth's theorem for K, in the signatures used here, states that for every sentence ϕ of L' in signature $\{e,r,p\}$, if for every (D,E,R) there is at most one $(D,E,R,P) \models \phi$, then there is a sentence ψ of L' and a formula θ with one free variable x_1 of L', both in signature $\{e,r\}$, such that $\phi \leftrightarrow (\psi \& (\forall x_1)(p(x_1) \leftrightarrow \theta))$ is valid.

We will make use of the fact that if Beth's theorem were true for such an L' then for every sentence ϕ of L' in signature $\{e,r,p,\}$ if for every (D,E,R) there is at most one $(D,E,R,P) \models \phi$, then $\{(D,E,R)$: there is a $(D,E,R,P) \models \phi\}$ is an elementary class in L^*.

In order to obtain the desired results mentioned in the introduction, it therefore suffices to find sentences ϕ_α such that

A) ϕ_0 is a sentence in $L(Q)$; if ω_α is singular then ϕ_α is a sentence in $L(Q_\alpha)$; if ω_α is regular, $\alpha > 0$, then ϕ_α is a sentence in $L^-(Q_\alpha)$

B) Each ϕ_α is of signature $\{e,r,p\}$, and for each (D,E,R) there is at most one $(D,E,R,P) \models \phi_\alpha$

C) Each $\{(D,E,R)$: there is a $(D,E,R,P) \models \phi_\alpha\}$ is not an elementary class in L^*.

Lipner [5] and Brown [2] developed back and forth criteria for elementary equivalence in languages with cardinality quantifiers.

A straightforward adaptation of their work will yield the back and forth criterion for $\equiv$ given below.

Let M, N be two structures in the same signature. A quasi-isomorphism from M onto N is a set K of finite partial isomorphisms from M into N that is closed under restrictions and is nonempty. If K is a quasi-isomorphism, then for each $f \in K$ we define the many-valued functions $K_f^1 : M \to N$ and $K_f^2 : N \to M$ by: $K_f^1(x) = y$ iff $f \cup \{\langle x,y \rangle\} \in K$, $K_f^2(x) = y$ iff $f \cup \{\langle y,x \rangle\} \in K$.

LEMMA 1. *Let M, N be structures in the same signature. Then $M \equiv N$ if and only if there exists a quasi-isomorphism K such that for all $f \in K$, the images of K_f^1, K_f^2 on any $x \subset M$, $y \subset N$ respectively have cardinality at least that of x, y respectively.*

2. The sentences

A double equivalence relation $(2 \sim r)$ is defined here as a structure (D,E,R) that (1) E is an equivalence relation on D, (2) $(R(a,b) \& E(b,c)) \to R(a,c)$, and (3)

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† Strictly speaking, assume that every formula of L' is a formula of L^*. Of course, a syntax free treatment can be given.

$(\exists b)\,(R(a,b))$. Thus a $2 \sim r$ is a (D,E,R) such that E is an equivalence relation on D and R associates to each $a \in D$ a nonempty set of equivalence classes under E.

Let (D,E,R) be a $2 \sim r$. The 0-small elements of (D,E,R) are those $x \in D$ such that E divides $\{a : R(x,a)\}$ into fewer equivalence classes than there are elements in D. The α-small elements, $\alpha > 0$, of (D,E,R) are those $x \in D$ such that E divides $\{a : R(x,a)\}$ into $< \omega_\alpha$ equivalence classes. The α-large elements are just those elements that are not α-small.

Let $\alpha = 0$ or ω_α singular. We say that (D,E,R) is an α-special $2 \sim r$ if and only if (D,E,R) is a $2 \sim r$ such that every equivalence class associated by R to an α-small $x \in D$ has exactly one α-small member, and every equivalence class associated by R to an α-large $x \in D$ has exactly one α-large member.

For regular $\omega_\alpha, \alpha > 0$, we say that (D,E,R) is α-special if and only if (D,E,R) is a $2 \sim r$ such that every equivalence class associated by R to an α-small $x \in D$ has at least one, but $< \omega_\alpha$ α-small members, and every equivalence class associated by R to an α-large $x \in D$ has at least one, but $< \omega_\alpha$ α-large members.

LEMMA 2. *For every $\alpha \geqq 0$ there is a sentence ϕ_α of signature $\{e,r,p\}$ such that* (a) $(D,E,R,P) \models \phi_\alpha$ *iff* (D,E,R) *is an α-special $2 \sim r$, and $P(x)$ iff x is α-small in (D,E,R), and* (b) *if $\alpha = 0$ then ϕ_α is in $L(Q)$; if ω_α is singular then ϕ_α is in $L(Q_\alpha)$; otherwise ϕ_α is in $L^{-}(Q_\alpha)$. Thus* (A) *and* (B) *of Section* 1 *are completed.*

PROOF. Define ϕ_0 to be the conjunction of the following: (i) axioms saying that (D,E,R) is a $2 \sim r$, (ii) $p(x) \to (\forall y)(r(x,y) \to (\exists! z)(e(y,z) \;\&\; p(z)))$, (iii) $\neg p(x) \to (\forall y)(r(x,y) \to (\exists! z)(e(y,z) \;\&\; \neg p(z)))$, (iv) $p(x) \to \neg (Qy)(p(y) \;\&\; r(x,y))$, and (v) $\neg p(x) \to (Qy)(\neg p(y) \;\&\; r(x,y))$.

Suppose (D,E,R) is a 0-special $2 \sim r$ with $P(x)$ iff x is 0-small in (D,E,R). Then obviously $(D,E,R,P) \models$ (i), (ii), (iii), and (iv) and (v) hold since E divides $\{y : r(x,y)\}$ into fewer equivalence classes than there are elements in D if and only if x is 0-small in (D,E,R).

Conversely, suppose $(D,E,R,P) \models \phi_0$. Then (D,E,R) is a $2 \sim r$. It is clear from (ii) and (iv) that $P(x)$ implies x is 0-small. It is equally clear from (iii) and (v) that $\neg P(x)$ implies x is 0-large. Hence $P(x)$ iff is 0-small in (D,E,R). Upon rereading (ii) and (iii), we see that (D,E,R) is 0-special.

Let ϕ_α, for ω_α singular, be the same as ϕ_0 except that Q is replaced by Q_α. Then precisely the same argument as above establishes that $(D,E,R,P) \models \phi_\alpha$ if and only if (D,E,R) is an α-special $2 \sim r$ and $P(x)$ iff x is α-small in (D,E,R).

Note that because of the exclamation marks in (ii) and (iii), we are using equality in ϕ_α for $\alpha = 0$ or ω_α singular. Define ϕ_α for $\alpha > 0$, ω_α regular, as the conjunction of the following: (1) Axioms saying that (D,E,R) is a $2 \sim r$, (2) $p(x) \rightarrow (\forall y)(r(x,y) \rightarrow \neg(Q_\alpha z)(e(y,z) \ \& \ p(z)))$, (3) $p(x) \rightarrow (\forall y)(r(x,y) \rightarrow (\exists z)(e(y,z) \ \& \ p(z)))$, (4) $\neg p(x) \rightarrow (\forall y)(r(x,y) \rightarrow \neg(Q_\alpha z)(e(y,z) \ \& \ p(z)))$, (5) $\neg p(x) \rightarrow (\forall y)(r(x,y) \rightarrow (\exists z)(e(y,z) \ \& \ \neg p(z)))$, (6) $p(x) \rightarrow \neg(Q_\alpha y)(p(y) \ \& \ r(x,y))$, and (7) $\neg p(x) \rightarrow (Q_\alpha y)(\neg p(y) \ \& \ r(x,y))$.

Suppose (D,E,R) is an α-special $2 \sim r$ with $P(x)$ iff x is α-small in (D,E,R), $\alpha > 0$, ω_α regular. Obviously $(D,E,R,P) \models$ (1), (2), (3), (4), and (5). By regularity of ω_α, we have $(D,E,R,P) \models$ (6) and (7).

Conversely, suppose $(D,E,R,P) \models \phi_\alpha$, $\alpha > 0$, ω_α regular. Then (D,E,R) is a $2 \sim r$. It is clear from (3) and (6) that $P(x)$ implies x is α-small. It is clear from (4) and (7) and the regularity of ω_α, that $\neg P(x)$ implies x is α-large. Hence $P(x)$ iff x is α-small. Upon rereading (2), (3), (4), and (5), we see that (D,E,R) is α-special.

3. α-Normal $2 \sim$ r's

For $\alpha > 0$, take D_α to be the set of all finite sequences of elements of ω_α. Take $D_0 = D_1$. Take R_α to be the binary relation on D_α given by $R_\alpha(f,g) \leftrightarrow (\exists n \in \omega)(\exists \beta)(g = f \cup \{\langle n,\beta\rangle\})$. An α-normal $2 \sim r$ is a $2 \sim r$ of the form (D_α,E,R_α), where E is an equivalence relation on D_α satisfying (i) $E(f,g) \rightarrow (\exists h)(R(h,f) \ \& \ R(h,g))$, and (ii) E divides each $\{g: (\exists h)(R(h,f) \ \& \ R(h,g))\}$ into infinitely many equivalence classes, each of power ω_α (power ω_1 if $\alpha = 0$).

LEMMA 3. *For each α, any two α-normal $2 \sim r$'s are* $\equiv$.

PROOF. We use the back and forth criterion of Lemma 1. Let $\alpha > 0$. Let $(D_\alpha,E_0,R_\alpha),(D_\alpha,E_1,R_\alpha)$ be two α-normal $2 \sim r$'s. We wish to prove $(D_\alpha,E_0,R_\alpha) \equiv (D_\alpha,E_1,R_\alpha)$. To do this, we first let K^+ be the set of all finite nonempty partial isomorphisms $\rho: (D_\alpha,E_0,R_\alpha) \rightarrow (D_\alpha,E_1,R_\alpha)$ such that (i) f and $\rho(f)$ have the same domain, and (ii) $\mathrm{Dom}(\rho)$ is closed under restrictions.

We first prove a very strong back and forth property for K^+. We show that if $\rho \in K^+$, $f \in D_\alpha - \mathrm{Dom}(\rho)$, then there are ω_α many f^* such that for some $\rho^* \in K^+$ extending ρ, $\rho^*(f) = f^*$.

Let $\rho \in K^+$, $f \in D_\alpha - \mathrm{Dom}(\rho)$. Choose h to be the smallest restriction of f not in Dom (ρ).

Case 1. For some $g \in \mathrm{Dom}(\rho)$, $E_0(h,g)$. Let $f^* \notin \mathrm{Rng}(\rho)$, $E_1(f^*,\rho(g))$. There are ω_α such f^*. For each choice of f^* choose $\rho^* \in K^+$ extending ρ, with $\rho^*(h) = f^*$.

Case 2. For no $g \in \mathrm{Dom}(\rho)$ is $E_0(h,g)$. Let $R_\alpha(h_0,h)$. Choose f^* with $R_\alpha(h_0,f^*)$ and for no $g \in \mathrm{Rng}(\rho)$ is $E_1(h,g)$. There are ω_α such f^* by clauses (iii) and (ii in the definitions of K^+, α-normal $2 \sim r$ respectively. For each choice of f^* choose $\rho^* \in K^+$ extending ρ, with $\rho^*(h) = f^*$.

Now let K be the set of all finite partial isomorphisms from (D_α,E_0,R_α) into (D_α,E_1,R_α) which have an extension in K^+. It is clear that K is a quasi-isomorphism. Let $\rho \in K, x \subset D_\alpha$. Let ρ^+ be any extension of ρ lying in K^+. If not $x \subset$ Dom (ρ^+), then by what we have proved about K^+, clearly the image of K^1_ρ under x has power ω_α. If $x \subset$ Dom (ρ^+), then since ρ^+ is one-one, clearly the image of K^1_ρ under x has power at least that of x.

By symmetry, we have shown the existence of a K satisfying Lemma 1. This therefore concludes the proof of Lemma 3.

We now just have to establish (C) of Section 1.

LEMMA 4. *For each α there is an α-normal $2 \sim r$ that is not α-special.*

PROOF. For such a α-normal $2 \sim r$, $\alpha > 0$, choose an α-normal $2 \sim r, (D_\alpha, E, R_\alpha)$, such that E divides each $\{g\colon (\exists h)(R(h,f) \,\&\, R(h,g))\}$ into ω_α many equivalence classes.

LEMMA 5. *For each $\alpha > 0$ there is a function $F\colon D_\alpha \to \{0,1\}$ satisfying* (1) $F(f) = 0$ *implies* $|\{g\colon R_\alpha(f,g) \,\&\, F(g) = 0\}| = \omega$, *and* (2) $F(f) = 1$ *implies* $|\{g\colon R_\alpha(f,g) \,\&\, F(g) = 0\}| = |\{g : R_\alpha(f,g) \,\&\, F(g) = 1\}| = \omega_\alpha$.

PROOF. The proof is a standard definition by recursion on the lengths (domains) of the elements of D_α.

LEMMA 6. *For each α there is an α-normal $2 \sim r$ which is α-special.*

PROOF. Let $\alpha > 0$. Choose F as in Lemma 5. If $F(f) = 0$ then define E on $\{g : R_\alpha(f,g)\}$ so that $\{g : R_\alpha(f,g)\}$ is divided into ω equivalence classes under E, each of which is of power ω_α and contains exactly one g with $F(t) = 0$. If $F(f) = 1$, then define E on $\{g : R_\alpha(f,g)\}$ so that $\{g : R_\alpha(f,g)\}$ is divided into ω_α equivalence classes under E, each of which is of power ω_α and contains exactly one g with $F(g) = 1$.

LEMMA 7. *For each* α, $\{(D,E,R):(D,E,R)$ *is an* α*-special* $2\sim r\}$ *is not an elementary class in* L^*.

PROOF. It is obvious from Lemmas 3, 4, and 6.

THEOREM. *Beth's theorem fails for any logic between* (1) $L(Q)$ *and* L^*, (2) $L(Q_\alpha)$ *and* L^*, *any* $\alpha > 0$, *and* (3) $L^-(Q_\alpha)$ *and* L^*, *if* ω_α *is regular*, $\alpha > 0$.

PROOF. By Lemmas 2 and 7, we have competed (A), (B), and (C) of Section 1. Hence, we are done.

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